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# Approximation of Euler-type equations by systems of vortices

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**Abstract.** We prove the weak convergence for any time of a system of quasi-vortex with positive and negative signs and without any truncature of the kernel to the solution of the Euler equation. Quasi-vortex means here that the kernel has a singularity in  $1/|x|^\alpha$  with  $\alpha \leq 1$  instead of diverging in  $1/|x|$  near the origin. We also give some bounds on the force field for the true vortex case and explain why our technic fails in this case.

**Key words.** Derivation of kinetic equations. Particle methods. Euler equation.

## 1 Introduction

Before making a review of know result, and introducing our result, we shall write the problem to fix the notation. We shall consider here system of  $N$  'generalized' vortex ( $N \in \mathbb{N}$ ) evolving in  $\mathbb{R}^2$ . We denote their position by  $(X_1, \dots, X_N)$  and the strength of the  $i$ -th vortex by  $\omega_i/N$  with  $\omega_i$  in  $[-1, 1]$ . These vortices are governed by the following system of differential equations:

$$\dot{X}_i = u(X_i(t)) = \frac{1}{N} \sum_{j \neq i}^N \omega_j K(X_i - X_j) \quad \text{for } i = 1, \dots, N, \quad (1.1)$$

where  $K : \mathbb{R} \rightarrow \mathbb{R}$  is the kernel of interaction. Here the kernel will be  $C^\infty$  except at the origin. So, once we precise the initial conditions  $(X_1^0, \dots, X_N^0)$ , for which we assume there are not two vortices at the same place, there is a unique solution defined till the first collision.

This equation was originally introduced in the case of the Biot-Savart kernel  $K(x) = x^\perp/|x|^2$  as a discrete approximation of the Euler equation written just above.

Here we shall focus on the limit of the distribution of the vortex  $\omega_N(t) = (1/N) \sum_{i=1}^N \omega_i \delta_{X_i(t)}$  when the number of particles increased until  $+\infty$ .

The continuous equation associated to that system (for one time, this is the discrete model that is derivated from the continuous one) is the Euler equation

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \\ u(x) = \int K(x - y) \omega(y) dy \end{cases} = K \star \omega \quad (1.2)$$

with a given initial vorticity  $\omega^0$ . Remark that the distribution of vorticity  $\omega_N(t)$  of the first system (1.1) solve the Euler equation in the sense of distribution if the kernel is regular.

What kinds of kernel  $K$  are usually used? For a system of vortex in dimension two,  $K$  is given by the Biot Savard law:

$$K(x) = \frac{x^\perp}{|x|^2}$$

where  $x^\perp$  is the vector of same length as  $x$  so that  $(x, x^\perp)$  is a direct orthogonal basis. This is the case of physical interest. However, in the first part of our article we will state theorems for forces satisfying

$$(C_\alpha) \quad |K(x)| \leq C \frac{1}{|x|^\alpha}, \quad |\nabla K(x)| \leq \frac{C}{|x|^{\alpha+1}}, \quad \text{div}(K) = 0, \text{ for some } \alpha < 1.$$

In this case, we will obtain estimates that will allow us to conclude that convergence occurs

Here, we will prove that,  $\omega_N(t)$  converge towards the unique solution of the Euler equation (1.2), if  $K$  comports itself like  $1/|x|^\alpha$ ,  $\alpha < 1$  in the neighbourhood of 0.

The subject of approximation of the Euler equation has been studied by many authors. The weak convergence for the vortex system was proven by Jean-Marc Delort in [Del91]. Steve Schochet simplified the proof in [Sch96] and [Sch95]. The principle of their demonstration is to lower the singularity with a symmetrisation of the kernel. This can only yield weak convergence. A work of great interest was done by J. Goodman, T.Y. Hou and J. Lowengrub, using numerical technics. They prove the convergence of the vortex method with well-distributed initial conditions to the Euler system [GHL90], [GH91]. The restriction of their work is that they use vortices initially distributed on a regular grid, but that allows them to obtain a good order of convergence, and without any truncation of the kernel. Here, we shall use vortices initially distributed with some regularity, but our assumptions are not so strong as those in [GHL90]. We will not obtain order of convergence, but the result of convergence is stronger than the one of S. Schochet [Sch95]. Our results will use the techniques introduced by Pierre-Emmanuel Jabin and the author in [MJ03].

## 1.2 Resolution of the ODEs

Before speaking of the limit when the number of particles became very large, we need to talk a little bit about the resolution of the system of ordinary equations. A good reference for this problem is the book of C. Marchioro and M. Pulvirenti [MP94]. They do a careful analysis of the system to show that the problem of the singularity can be solved and that the set of the initial conditions for which collision occurs is negligible. Here, thanks to the hypotheses we will put on the initial conditions, we will show that collisions never occur for the system of similit-vortex system, and that there are not any collisions till a time  $T$  for the system of true vortices. We also refer to [MP94] for the existence and the uniqueness of solution of the Euler equation when the vorticity is in  $L^\infty$ .

We will show some result of convergence of system of vortex to the continuous equation it approximates, the Euler equation.

## 2 Main results

To state our result, we introduce a notion of  $L^\infty$  discrete norm.

**Definition 1** ( $L^\infty$  discrete norm). *Choose an  $\eta$  in  $\mathbb{R}$ , an a signed measure  $\omega$  in  $\mathbb{R}^2$ , the  $L^\infty$ -discrete norm of  $\mu$  at scale  $\eta$  is*

$$\|\omega\|_{\infty, \eta} = \sup_{x \in \mathbb{R}^2} \frac{1}{\eta^2} |\omega|(B(x, \eta)) = \frac{\|\omega| \star \xi_{B(0, \eta/2)}\|_\infty}{\text{Vol}(B(0, \eta/2))}$$

where  $B(0, \eta)$  denotes the ball of center 0 and diameter  $\eta$ ,  $|\omega|(A)$  is the total variation of  $\omega$  in the set  $A$ ,  $\xi_{B(0, \eta)}$ , the characteristic function of the ball of radius  $\eta/2$  and  $\text{Vol}$  is the Lebesgue measure on  $\mathbb{R}^2$ .

**Remark 1.** *This is the definition for  $\mathbb{R}^2$  because we will only work on  $\mathbb{R}^2$ , but this can of course be extended for all the  $\mathbb{R}^n$ . If  $\omega$  is a sum of  $N$  vortex of strength  $\pm 1/N$ ,  $\|\omega\|_{\infty, \eta}$  compute the number of vortex in a ball of size  $\eta$ .*

Using this norm is like saying that we do not want to look too close at the vortices. If we know their position with an uncertainty of  $\eta$ , it will be sufficient for us. It allows us to get rid of the singularity of the Dirac masses. But on the other hand, we have to prove that the uncertainty on the position of the Dirac masses will not have any consequences in the calculation.

We want to look for every  $N$  at a scale that goes to zero when  $\varepsilon$  goes to infinity, because at the limit a uniform bound on such  $L^\infty$ -discrete norm will give us true  $L^\infty$  bounds. Now, at which scale  $\eta$  can we look at? We have vortex of strength  $1/N$ . We define  $\varepsilon$  by  $N = \varepsilon^{-2}$ . Remark that since we have  $\varepsilon^{-2}$  vortex in dimension 2,  $\varepsilon$  is the order of the average distance between vortex. We want to look at scale of type  $\varepsilon^\kappa$ . If  $\kappa$  is strictly greater than 1, the initial  $\varepsilon^\kappa$  norm of the distribution of vortex automatically blows up as  $\varepsilon \rightarrow 0$ . So  $\varepsilon$  is the smallest scale we can look at, our microscopical scale. At this scale and roughly speaking, we deal with a finite number of particles per box. If we look at a scale  $\varepsilon^\kappa$  with a  $\kappa \in (0, 1)$ , the number of particles by box goes to infinity and maybe we will be able to observe macroscopical comportement at this scale. The use of that norm at different scales will be crucial in the rest of this article.

We will use this tool in the case of the similit-vortex. By this we mean the system of ODE when the force satisfy the condition  $(C_\alpha)$  that we recall here:

$$(C_\alpha) \quad |K(x)| \leq C \frac{1}{|x|^\alpha}, \quad |\nabla K(x)| \leq \frac{C}{|x|^{\alpha+1}}, \quad \text{div}(K) = 0, \text{ for some } \alpha < 1.$$

We first define two quantities.

-  $R(t)$  is the size of the support of  $\mu_N$ :

$$R(t) = \sup_{i=1,\dots,N} |X_i(t)|.$$

-  $m(t)$  control the minimal distance between two vortices.

$$m(t) = \sup_{i \neq j} \frac{\varepsilon}{|X_i(t) - X_j(t)|}$$

We recall that we define the distribution of vorticity  $\omega_N$  by

$$\omega_N(t) = \frac{1}{N} \sum_{i=1}^N \omega_i \delta_{X_i(t)}$$

For this system, we obtain the following result of convergence:

**Theorem 1.** *For each  $N$ , choose the initial positions of  $N$  vortices so that  $R(0)$ ,  $m(0)$  and  $\|\omega_N\|_{\infty,\varepsilon}$  are uniformly bounded. Moreover, assume that  $\mu_N(0)$  goes weakly in the sense of measure to a function  $\omega^0$ , which is then in  $L^\infty$ . Then, for every  $t$  in  $\mathbb{R}$ ,  $\omega_N(t)$  goes weakly to  $\omega(t)$ , the unique solution of the Euler equation (1.2) with initial conditions  $\omega^0$ .*

**Remark 2.** *We were not able to prove a same result for the true vortex case, in the last section we explain why. We will obtain some optimal bound which will not allows us to conclude the proof of convergence.*

**Warnings.** We will often erase the time  $t$  or the subscript  $N$  in our calculation, but the reader should keep in mind that we always do calculations at a fixed time  $t$  and for a fixed  $N$ . We will also use many time  $C$  as a numerical constant whose value can change from one line to another. An equation with a  $C$  in it means that there exists a numerical constant (a real) so that... Non-numerical constants will be denoted by other letters as  $K_i$ ...

### 3 The system of simili-vortices

This section is devoted to the proof of Theorem 1. Before beginning the proof, we will give a sketch of it.

#### 3.1 Sketch of the proof.

Our is to show that a norm  $\|\omega_N t\|_{\infty,\eta}$  remains bounded independantly of  $N$ . Once we get that bound, the convergence follows (see Lemma 3 and the following). For this, we give bounds of the speeds of the vortex (called  $U$ ) and of a discrete equivalent of the derivative of the speed ( $\nabla U$ ), using discrete equivalent of Hölder inequalities. This bound allows us to obtain differential inequalities satisfied by the couple  $(R_N(t), m_N(t))$ , independant of  $N$  except for small terms. These differential inequalities implies the existence a time  $T^*$  till which these quantities do not explode, (and so do  $\|\omega_N t\|_{\infty,\varepsilon}$ ), independantly of  $N$ . This gives the convergence for short time. To obtain long-time convergence, we replace the system of differential inequalities in  $(R, m)$ , by a new one, sub-linear, which never explodes. To get this new system, we get more accurate bound on  $U$  and  $\nabla U$ , introducing new scales  $\eta$  greater that  $\varepsilon$ , to show that at this scales, the  $\|\omega_N t\|_{\infty,\eta}$  is almost preserved.

According to the sketch, we will begin by the convergence for short time.

#### 3.2 Convergence for short time

*Step 1. Estimate of the speeds.* We define

$$U(t) = \sup_{i=1,\dots,N} u(X_i(t)) = \sup \left( \frac{1}{N} \sum_{j \neq i}^N \omega_j K(X_i - X_j) \right)$$

Then we have the following result

**Lemma 1.** *We have the following bound*

$$U \leq C \|\omega\|_{\infty,\varepsilon}^{1/2} R^{1-\alpha} + C \varepsilon^{2-\alpha} \|\omega\|_{\infty,\varepsilon} m(t)^\alpha$$

*Proof of the lemma.* In the continuous case, estimate of this type are obtain thanks to Hölder inequality. We will use a discrete equivalent of this inequality. We choose one  $i$  and decompose the space  $\mathbb{R}^2$  in the subset  $I_0 = \{|x - X_i| \leq \varepsilon\}$  and  $I_k = \{|x - X_i| \in [2^k \varepsilon, 2^{k+1} \varepsilon)\}$  for all  $k \in \mathbb{N}$ . The greatest  $k$  such that  $I_k$  may contain at least one particle is  $k_{max} = \lfloor R/\varepsilon \rfloor + 1$ , where the brackets denoted the integer part. We denote by  $u_k$  the part of the speed of  $X_i$  due to the vortices in  $I_k$ . It is easy to bound  $u_0$  by

$$|u_0| \leq \|\omega\|_{\infty, \varepsilon} \varepsilon^2 \left( \frac{m(t)}{\varepsilon} \right)^\alpha = \|\omega\|_{\infty, \varepsilon} \varepsilon^{2-\alpha} m(t)^\alpha$$

using the lowest bound  $\varepsilon/m(t)$  for the minimal inter-vortices distance. For the remaining terms, we use a discrete analog of an Hölder inequality. For this, remark that

$$|\omega|(I_k) \leq 1 \quad \text{and that} \quad |\omega|(I_k) \leq \|\omega\|_{\infty, \varepsilon} (2^{k+1} \varepsilon)^2$$

If we take the first inequality at the power  $1/2$  multiplied by the second at the power  $1/2$ , we obtain

$$|\omega|(I_k) \leq 4 \|\omega\|_{\infty, \varepsilon}^{1/2} 2^{k+1} \varepsilon$$

Moreover, for each  $X_j \in I_k$ , we have  $|X_j - X_i| \geq 2^k \varepsilon$ . So we can bound  $|u_k|$  by  $|u_k| \leq 8 \|\omega\|_{\infty, \varepsilon}^{1/2} (2^{k+1} \varepsilon)^{1-\alpha}$ . Now, we sum from  $k = 0$  to  $k_{max}$ . We obtain

$$\sum_{k=0}^{k_{max}} |u_k| \leq 16 \|\omega\|_{\infty, \varepsilon}^{1/2} \varepsilon^{1-\alpha} \sum_{k=1}^{k_{max}} 2^{k(1-\alpha)} \quad (3.1)$$

$$\leq C \|\omega\|_{\infty, \varepsilon}^{1/2} R^{1-\alpha} \quad (3.2)$$

Adding the bound on  $u_0$  to this one give the expected result.  $\square$

*Step 2. Estimation of the derivative of the speed.*

We define

$$\bar{\nabla} U(t) = \sup_{i \neq j} \frac{|u(X_i(t)) - u(X_j(t))|}{|X_j(t) - X_i(t)|}$$

We use this definition instead of a true bound on  $\bar{\nabla} U = \omega_N * \nabla K$  because a vortex does not interact with itself and then this quantity is not of real interest. The following lemma will give us a bound on  $\bar{\nabla} U(t)$

**Lemma 2.** *We have the following bound*

$$\bar{\nabla} U(t) \leq C \|\omega(t)\|_{\infty, \varepsilon} R(t)^{1-\alpha} + C \varepsilon^{1-\alpha} (1 + \|\omega(t)\|_{\infty, \varepsilon}) m(t)^{1+\alpha}$$

*Proof.* We pick an  $i$  and a  $j$ . Then,

$$\begin{aligned} \frac{|u(X_i) - u(X_j)|}{|X_j - X_i|} &\leq \frac{|K(X_i - X_j) - K(X_j - X_i)|}{N |X_i - X_j|} \\ &\quad + \frac{1}{|X_j - X_i|} \sum_{k \neq i, j} |K(X_k - X_j) - K(X_k - X_i)| \end{aligned} \quad (3.3)$$

The first term is bounded by  $C \varepsilon^2 |X_i - X_j|^{-(1+\alpha)} \leq C \varepsilon^{1-\alpha} m(t)^{1+\alpha}$ . To bound the others, we use

$$\frac{|K(X_k - X_j) - K(X_k - X_i)|}{|X_j - X_i|} \leq \frac{C}{\min(|X_k - X_i|, |X_k - X_j|)^{1+\alpha}}.$$

This inequality come from the condition  $(C_\alpha)$  on the derivative of  $K$ . We decompose the space into the subsets  $I_l = \{x | \min(|x - X_i|, |x - X_j|) \in [l\varepsilon, (l+1)\varepsilon)\}$  for  $l$  equals 0 to  $l_{max} = \lfloor R/2\varepsilon \rfloor$ . Remark that we do not use the same decomposition that in the proof of the estimate for the speed field, because we use the discrete Hölder inequality with  $p = 1$  and  $q = \infty$ . The absolute vorticity in  $I_l$  is bounded by

$$|\omega|(I_l) \leq C \|\omega\|_{\infty, \varepsilon} l \varepsilon^2,$$

and for every vortex  $k$  in  $I_l$ ,  $\min(|X_k - X_j|, |X_k - X_i|) \geq l\varepsilon$ . Then,

$$\frac{1}{N} \sum_{\omega_i \in I_l} \frac{|K(X_k - X_j) - K(X_k - X_i)|}{|X_j - X_i|} \leq C \|\omega\|_{\infty, \varepsilon} \varepsilon^{1-\alpha} l^{-\alpha}.$$

Separately, the contribution of  $I_0$  can be bounded by  $C \|\omega\|_{\infty, \varepsilon} \varepsilon^{1-\alpha} m(t)^{1+\alpha}$ . Adding all these contributions, we get

$$\bar{\nabla} U(t) \leq C \|\omega\|_{\infty, \varepsilon} \varepsilon^{1-\alpha} \sum_{l=1}^{l_{max}} l^{-\alpha} + C (\|\omega\|_{\infty, \varepsilon} + 1) \varepsilon^{1-\alpha} m(t)^{1+\alpha}$$

We may bound the sum  $\sum_{l=1}^{l_{max}} l^{-\alpha}$  by  $C l_{max}^{1-\alpha} = C R^{1-\alpha} \varepsilon^{\alpha-1}$ . This gives the expected result.  $\square$

Step 3. A system of differential inequalities.

Now we want to control the growth of  $R$  and  $m$ . For  $R$ , we can obtain

$$\dot{R}(t) = \frac{\partial}{\partial t} \left( \sup_{i \leq N} X_i(t) \right) \quad (3.4)$$

$$\leq \sup \left( \frac{\partial}{\partial t} |X_i(t)| \right) \quad (3.5)$$

$$\leq \sup |\dot{X}_i(t)| \leq U(t). \quad (3.6)$$

For  $m$ , we do as above:

$$\dot{m}(t) = \frac{\partial}{\partial t} \left( \sup_{i \neq j} \frac{\varepsilon}{|X_i(t) - X_j(t)|} \right) \quad (3.7)$$

$$\leq \sup_{i \neq j} \frac{C\varepsilon |U(X_i(t)) - U(X_j(t))|}{|X_i(t) - X_j(t)|^2} \quad (3.8)$$

$$\leq m(t) \nabla U(t). \quad (3.9)$$

We finally obtain the following system of ordinary differential equations:

$$\begin{cases} \dot{m}(t) \leq \nabla U(t) m(t) \\ \dot{R}(t) \leq U(t) \end{cases} \quad (3.10)$$

With this and the bounds (3.2) and (2), we can bound  $\dot{m}$  and  $\dot{R}$  in function of  $m$ ,  $R$ , and  $\|\omega\|_{\infty, \varepsilon}$ . Moreover, it is possible to bound  $\|\omega\|_{\infty, \varepsilon}$  in terms of  $m(t)$ . Indeed, we can not put more than  $CM^2$  particles in a ball of size  $\varepsilon$  if we want that the minimal distance between particles to be greater than  $\varepsilon/M$ . So, we can bound  $\|\omega\|_{\infty, \varepsilon}$  by  $\|\omega\|_{\infty, \varepsilon} \leq Cm(t)^2$ . Putting all together, we get the following system of differential inequalities for  $m$  and  $R$ :

$$\begin{cases} \dot{m}(t) \leq Cm(t)^3 R(t)^{1-\alpha} + C\varepsilon^{1-\alpha} m(t)^{4+\alpha} \\ \dot{R}(t) \leq Cm(t)^2 R(t)^{1-\alpha} + C\varepsilon^{2-\alpha} m(t)^{2+\alpha} \end{cases} \quad (3.11)$$

As long as the two quantities containing  $\varepsilon$ , that is  $C\varepsilon^{1-\alpha} m(t)^{4+\alpha}$  and  $C\varepsilon^{2-\alpha} m(t)^{2+\alpha}$  are less than one, a condition that is true at time  $t = 0$  if  $\varepsilon$  is sufficiently small, we may write

$$\begin{cases} \dot{m}(t) \leq Cm(t)^3 R(t)^{1-\alpha} + 1 \\ \dot{R}(t) \leq Cm(t) R(t)^{1-\alpha} + 1 \end{cases} \quad (3.12)$$

Now, we choose  $m_0$  and  $R_0$  so that  $m_N(0) \leq m_0$  and  $R_N(0) \leq R_0$  for all  $N$ . We denote also  $(m_t, R_t)$  the solution of the ODE

$$\begin{cases} \dot{m}(t) = Cm(t)^3 R(t)^{1-\alpha} + 1 \\ \dot{R}(t) = Cm(t) R(t)^{1-\alpha} + 1 \end{cases}$$

with initial conditions  $(m_0, R_0)$ . It exists till a time of explosion  $T^*$ . Since the right hand side terms are increasing in  $R$  and  $m$ , we can write  $m_N(t) \leq m_t$  and  $R_N(t) \leq R_t$  provided that the conditions  $C\varepsilon^{1-\alpha} m(t)^{4+\alpha} \leq 1$  and  $C\varepsilon^{2-\alpha} m(t)^{2+\alpha} \leq 1$  are true. This will be the case for any time  $t$  less than  $T^*$  if  $\varepsilon$  is small enough. So, we get uniform bound on  $m_N$  and  $R_N$  for  $t \leq T^*$ .

*Step 4. Conclusion of the convergence.*

So, we have uniform bounds on  $\|\omega(t)\|_{\infty, \varepsilon}$ ,  $m_N(t)$  and  $R_N(t)$  for all  $t < T^*$ . This will imply strong convergence results for the field of speeds, and allow us to take the limit in the equation. First, if we take a subsequence of  $\omega_N(t)$  that goes weakly in the sense of measure to  $\omega$ , this  $\omega$  belongs to  $L^\infty$ . This is proved in the following lemma:

**Lemma 3.** *Take a sequence of probability measure  $\omega_n$  on  $\mathbb{R}^2$  that converge weakly to  $\omega$ , and such that there exists a sequence  $\rho_n$  of positive real going to zero so that  $\|\omega_n\|_{\infty, \rho_n}$  is uniformly bounded. Then,  $\omega$  belongs to  $L^\infty$ .*

*Proof of the lemma.* We denote by  $\xi_n$  the characteristic function of the ball  $B(0, \rho_n)$  for the sup distance, divided by its volume  $4\rho_n^2$ :  $\xi_n = 1/(4\rho_n^2) \chi_{B(0, \rho_n)}$ . We choose a smooth test function  $\phi$ . We have

$$\int \phi(x) d\omega_n(x) = \int \phi(x) d(\omega_n - \omega_n * \xi_n)(x) + \int \phi(x) d(\omega_n * \xi_n)(x) \quad (3.13)$$

$$= \int (\phi(x) - \phi * \xi_n(x)) d\omega_n(x) + \int \phi(x) d(\omega_n * \xi_n)(x) \quad (3.14)$$

The first integral is bounded by  $\|\nabla\phi\|_\infty\rho_n$  by a classical estimate on convolution because  $\omega_n$  is of total mass one, and the second is bounded by  $\|\phi\|_1\|\omega_n\star\xi_n\|_\infty=\|\phi\|_1\|\omega_n\|_{\infty,\rho_n}$ . We get,

$$\left|\int\phi(x)d\omega_n(x)\right|\leq\|\nabla\phi\|_\infty\rho_n+\|\phi\|_1\|\omega_n\|_{\infty,\rho_n}$$

Taking the limit when  $n$  goes to  $+\infty$ , we obtain

$$\left|\int\phi(x)d\omega_n(x)\right|\leq\liminf_{n\rightarrow\infty}\|\phi\|_1\|\omega_n\|_{\infty,\rho_n}$$

Since this is true for every smooth  $\phi$ , this means that  $\omega$  belongs to  $L^\infty$  and that:

$$\|\omega\|_\infty\leq\liminf_{n\rightarrow\infty}\|\omega_n\|_{\infty,\rho_n}$$

□

Moreover, with those uniform bounds, we can obtain the strong convergence for the speed field. This is state in the following lemma

**Lemma 4.** *Choose a sequence  $\omega_n = 1/N \sum_{i=1}^n \delta_{X_i(t)}$  with the  $X_i$  solution of (1.1 such that  $\omega_n(0) \rightharpoonup \omega_0$ , for large  $n$ . Assume that there exists a sequence  $\rho_n$  of positive real going to zero and a time  $T$  such that  $\sup_{t\leq T}\|\omega_n\|_{\infty,\rho_n}$  is uniformly bounded. Then, the sequence  $\omega_N$  converge weakly to the solution  $\omega \in L^\infty$  of the Euler equation (1.2) with initial condition  $\omega_0$ .*

*Proof of the lemma.* We extract a subsequence in the  $\omega_N$  that we will still denote by  $\omega_N$  which converges to a  $\omega$  in  $L^\infty$ . We denote by  $u_N$  the speed field created by  $\omega_N$  and by  $u$  the field created by  $\omega$ . That means

$$u_N(x) = \int K(x-y)d\omega_N(x) \quad u(x) = \int K(x-y)d\omega(x),$$

with the convention  $K(0) = 0$ , because there is no self-interaction for the vortices. Then, for every positive  $r$  (and every time  $t \leq T$ )

$$\begin{aligned} |u_N(x) - u(x)| &= \left| \int K(x-y)d(\omega_N - \omega)(y) \right| \\ &\leq \int_{|x-y|\geq r} |K(x-y)|d(\omega_N - \omega)(y) + \int_{|x-y|\leq r} |K(x-y)|d(\omega_N + \omega)(y) \end{aligned}$$

The first term goes to 0 when  $N$  goes to  $+\infty$  because of the weak convergence of  $\omega_N$  to  $\omega$  and the continuity of the kernel outside the origin. The second can be bounded by  $(\|\omega\|_\infty + C\|\omega_N\|_{\infty,\varepsilon})r^{2-\alpha} + \varepsilon^{2-\alpha}\|\omega_N\|_{\infty,\varepsilon}m(t)^\alpha$  using the previous decomposition for the term due to  $\omega_N$  or equivalently by replacing  $R(t)$  by  $r$  in the proof of Lemma 1. So this term goes to zero when  $r$  goes to zero and we get the pointwise convergence of  $u_N$  to  $u$ . Moreover, the sequence  $u_N$  is uniformly bounded in  $L^\infty$ . This allows us to pass to the limit in the Euler equation, satisfied by all the  $\omega_N$  and we obtain that  $\omega$  is also a solution of the Euler equation, with initial conditions  $\omega_0$ . Since the solution of the Euler equation is unique in  $L^\infty$ , we get that the whole sequence  $\omega_N$  goes weakly to  $\omega$ , the solution of the Euler equation. And this gives us the convergence till the a time  $T^*$ . □

### 3.3 Long time convergence

How could we get a convergence for long time? For this, we need bounds on  $R$  and  $m$  for any time. This could be done if our system of inequalities were sub-linear. Remark that, it could be so if we could write that  $\|\omega(t)\|_{\infty,\varepsilon} = \|\omega\|_{\infty,\varepsilon}$ . In this case, we won't have to replace  $\|\omega\|_{\infty,\varepsilon}$  by  $m(t)^2$  in the sytem (3.10), and instead of (3.12), we will obtain a system of the form below:

$$\begin{cases} \dot{m}(t) \leq C\|\omega\|_{\infty,\varepsilon}m(t)R(t)^{1-\alpha} + 1 \\ \dot{R}(t) \leq C\|\omega\|_{\infty,\varepsilon}R(t)^{1-\alpha} + 1 \end{cases}$$

This system do not explode in a finite time. The second line give us a polynomial growth for  $R$  and once we obtain this growth, we obtain an exponential growth for  $m$  by replacing  $R$  by its bound in the first line. So we will get bound for our two quantities for every time. Remark also that this preservation of the  $L^\infty$  norm is obvious in the continuous model, the Euler equation and that here we will need  $\varepsilon$  to obtain. But how can we obtain bound of  $L^\infty$  discrete norms. It seems that the  $L^\infty$  norm at scale  $\varepsilon$  is not preserved. But, the answer is to look at larger scale, a mesoscopic scale. We will be able to obtain the asymptotic preservation of the discrete  $L^\infty$  norm at a mesoscopic level. The following proposition states it more precisely:

**Proposition 1.** Set  $T_1$  to be the time so that  $\int_0^{T_1} \|\omega_N(t)\|_{\infty} dt = 1/4$  and fix a  $t \leq T_1$ . Choose a  $\gamma \in (0, 1)$ . Then, there exist two constants  $K_1$  and  $K_2$  depending on  $m, R$  such that for all  $\eta \geq \varepsilon$

$$\|\omega_N(t)\|_{\infty, \eta} \leq (1 + K_2 \eta^\gamma + K_1 \frac{\varepsilon}{\eta}) \|\omega_N(0)\|_{\infty, \varepsilon}$$

To prove this proposition, we first introduce the following definitions:

**Definition 2** (Parallelogram). A parallelogram in  $\mathbb{R}^2$  is a set  $S$  defined by

$$S = \{x \mid \|A(x - x_0)\| \leq \rho\}$$

where  $A$  is a  $2 \times 2$  matrix of determinant 1,  $\rho$  is a positive real and  $x_0$  belongs to  $\mathbb{R}^2$ .  $A$ ,  $\rho$ ,  $x_0$  will be called respectively the matrix, the size and the center of the parallelogram.

Remark that this definition correspond to the usual definition, because we use the sup distance. Our parallelogram are just centered at the origin.

**Definition 3** (Not too stretched parallelogram). A parallelogram is not too stretched if

$$\|A - Id\| = \sup_{\|x\|=1} \|Ax - x\| \leq 1/2.$$

**Remark 3.** Roughly speaking, this definition means that our parallelogram has a shape close from the one of a square. It is not too stretched in one direction.

At a time  $t$ , we will look at a box  $S_t = \{x \mid |x - x_0| \leq \rho\}$  and let it evolves backward according to the field of velocity  $u_N$  created by the vortices till time  $t' \leq t$  not too far from  $t$ . We obtain a set denoted  $\tilde{S}_{t'}$ . We will show that this set could be included in a not too stretched parallelogram with almost the same volume than the initial box. Thank of the control on the shape of the parallelogram, we will be able to cover it by  $\varepsilon$ -ball, and control the number in particle in it. It is at this stage that we need this definition of not too stretched parallelogram. Because, to cover a rectangle of width  $\varepsilon^2$  and length 1 with  $\varepsilon$ -ball, we need much more ball than for a square of same volume. We will now introduce a notion of  $\varepsilon$ -volume and a lemma relating  $\varepsilon$ -volume and volume for not too stretched parallelogram.

**Definition 4** ( $\varepsilon$ -volume). The  $\varepsilon$ -volume of a set  $S$ , denoted by  $Vol_\varepsilon(S)$  is the minimal number of balls of diameter  $\varepsilon$  that we need to cover it, divided by the volume of such a ball (which equals the volume of the covering).

$$Vol_\varepsilon(S) = \inf\{Vol(\cup_i B(x_i, \varepsilon/2)) \mid N \in \mathbb{N}, x_1, \dots, x_n \in \mathbb{R}^2 \text{ so that } S \subset \cup_i B(x_i, \varepsilon/2)\}, \quad (3.15)$$

where  $Vol$  denote the Lebesgue measure of  $\mathbb{R}^2$ .

**Lemma 5.** Let  $S$  be a not too stretched parallelogram of size  $\rho$ . Then

$$Vol_\varepsilon(S) \leq (1 + 2\frac{\varepsilon}{\rho})^2 Vol(S)$$

*Proof of the lemma.* We suppose that  $x_0 = 0$  for simplicity. We set  $S_{+\varepsilon} = \{x \text{ so that } |Ax| \leq \rho + \varepsilon\}$  and  $S_{+2\varepsilon} = \{x \text{ so that } |Ax| \leq \rho + 2\varepsilon\}$ . We define  $P = S_{+\varepsilon} \cap \varepsilon\mathbb{Z}^2$  and  $P_\varepsilon = P + B(0, \varepsilon/2)$ , the union of all the balls of diameter  $\varepsilon/2$  with center in  $P$ .

First, we will show that  $S \subset P_\varepsilon$ . For this, we choose an  $x$  in  $S$  and associate to it a couple  $m$  of  $\mathbb{Z}^2$  so that  $|x - \varepsilon m| \leq \varepsilon/2$ . Then,

$$|A(\varepsilon m)| \leq |A(\varepsilon m - x)| + |Ax| \leq \|A\| |x - \varepsilon m| + \rho.$$

Since  $\|A\| \leq 2$ , we obtain that  $\varepsilon m$  belongs to  $S_{+\varepsilon}$ , and then to  $P$ .

Next, we shall show that  $P_\varepsilon \subset S_{+2\varepsilon}$ . We choose a  $y$  in  $P_\varepsilon$  and associate to it a couple  $m$  of  $\mathbb{Z}^2$  so that  $|y - \varepsilon m| \leq \varepsilon/2$ . Then,

$$|Ay| \leq |A(y - \varepsilon m)| + \varepsilon |Am| \leq \|A\| |y - \varepsilon m| + \rho + \varepsilon$$

Again, we obtain  $|Ay| \leq \rho + 2\varepsilon$  and then  $y$  belongs to  $S_{+2\varepsilon}$ . Now, we can compare the  $\varepsilon$ -volume of  $S$  and the volume of the two others sets. We have

$$Vol_\varepsilon(S) \leq Vol(P_\varepsilon) \leq Vol(S_{+2\varepsilon})$$

And  $Vol(S_{+2\varepsilon}) = (\det(A))^{-1}(\rho + 2\varepsilon)^2 = (1 + 2\frac{\varepsilon}{\rho})^2 Vol(S)$ . This concludes the proof.  $\square$



**Remark 4.** We also remark that the norm of the matrix of a not too stretched parallelogram and of its inverse are always less than 2 because  $\|A - Id\| \leq 1/2$ . The inequality on the inverse implies that a not too stretched parallelogram of center  $x_0$  is always included in the ball centered at  $x_0$  of radius twice its size.

With these tools, we shall control the backward evolution of the vortices that are in a parallelogram at time  $t$ . Before stating a lemma, we introduce some usefull notations. We will denote parallelogram by  $S_t$ , it means that it is related to the time  $t$ . And then, its size, center, and matrix will always be denoted by  $\rho_t$ ,  $x_t$  and  $A_t$ . We will also use an approximation of the field of the form

$$u_\varepsilon(x) = \sum_{i=1}^N \omega_i K_\varepsilon(x - X_i(t)), \quad (3.16)$$

where  $K_\varepsilon$  is an approximation of  $K$  given by  $K_\varepsilon = K \star \xi_\varepsilon$ .  $\xi_\varepsilon$  is a classical approximation of the identity, by instance  $\xi_\varepsilon = (1/\varepsilon^2)\xi(\cdot/\varepsilon)$  with a  $\xi \in C^\infty$  with support in  $B(0, 1)$  of total mass 1. This approximated field  $u_\varepsilon$  satisfy the same estimate than  $u$ , because  $K_\varepsilon$  satisfies the same conditions (1) than  $K$ .

**Lemma 6.** Choose a time  $t > 0$  and a  $\gamma \in (0, 1)$ . There exists two positive constants  $K_1$  and  $K_2$ , such that for every not too stretched parallelogram  $S_t$  (with size  $\rho_t$ , center  $x_t$  and matrice  $A_t$ ), there exists a time  $t^* < t$  and a family of not too stretched parallelogram  $(S_{t'})_{t^* < t' < t}$  (with size  $\rho_{t'}$ , center  $x_{t'}$  and matrice  $A_{t'}$ ) such all the vortices that are in  $S_t$  at time  $t$  are in  $S_{t'}$  at time  $t'$ . The parallelograms  $S_{t'}$  satisfy:

- i. their center  $x_{t'}$  is the point  $x_t$  transported backward in time by  $u_\varepsilon$ ,
- ii. their matrix  $A_{t'}$  are always of determinant 1 and satisfy the ODE

$$\dot{A}_s = -A_s \nabla u_\varepsilon(x_s)$$

- iii. their size  $\rho_{t'}$  satisfies

$$\dot{\rho}_s = -K_1 \varepsilon - K_2 \rho_s^{1+\gamma}.$$

The time  $t^*$  before which no control by a not too stretched parallelogram is possible is the time when  $\|A_t - Id\|$  becomes greater than  $1/2$  and is of the order of  $(1/2 - \|A_t - Id\|)/\nabla U(t)$ .

*Proof of the lemma.* We lemma means roughly that close vortices have a commun motion. In first approximation they all move according almost smooth flow they created. To show this precisely, like in this lemma we use approximation of the field do get ride of the singularity. We shall then control the difference between the approximation and the true field for a particle.

$$|u(X_i) - u_\varepsilon(X_i)| \leq \frac{1}{N} \sum_{j \neq i} |K(X_j - X_i) - K_\varepsilon(X_j - X_i)|.$$

Using the derivative of  $K$ , we can show that  $|K(x) - K_\varepsilon(x)| \leq \frac{C\varepsilon}{|x|^{1+\alpha}}$  Using this bound and our usual division of the space, we can compute the difference due to the particles at distance greater than  $\varepsilon$ . For the rest, we do not use the last bound and bound the difference by the sum of the two terms and bound it as above:

$$|K(x) - K_\varepsilon(x)| \leq \frac{C}{|x|^\alpha}$$

We obtain at the end

$$\sup_{i=1, \dots, N} |u(X_i) - u_\varepsilon(X_i)| \leq C\|\omega\|_{\infty, \varepsilon} R(t)^{1-\alpha} \varepsilon + C\varepsilon^{2-\alpha} \|\omega\|_{\infty, \varepsilon} m(t)^\alpha$$

So if we define  $K_1 = C\|\omega\|_{\infty, \varepsilon} R(t)^{1-\alpha} + m(t)^\alpha$ , we obtain

$$\sup_{i=1, \dots, N} |u(X_i) - u_\varepsilon(X_i)| \leq K_1 \varepsilon.$$

Now, to obtain the EDO on the size of the parallelogram, we shall bound the derivative with respect to the time of  $|A_t(X_i(t) - x_t)|$ . We have (remember that we are interested in backward estimates)

$$\begin{aligned} \frac{d}{dt} |A_t(X_i(t) - x_t)| &\geq -|A_t(-\nabla u_\varepsilon(x_s))(X_i(t) - x_t + u(X_i(t)) - u_\varepsilon(x_t))| \\ &\geq -\|A_t\| |u(X_i(t)) - u_\varepsilon(X_i(t))| - \dots \\ &\dots \|A_t\| |u_\varepsilon(X_i(t)) - u_\varepsilon(x_t) - \nabla u_\varepsilon(x_t)(X_i(t) - x_t)| \end{aligned} \quad (3.17)$$

The first contribution  $A_1$  is controlled using the bound (3.9) just above by  $\|A_t\|K_1\varepsilon$ . The second term  $A_2$  is the error between the field  $u_\varepsilon$  and its linearization near  $x_s$ . To bound this term, we remark that

$$|K_\varepsilon(x) - K_\varepsilon(y) - \nabla K_\varepsilon(y)(x - y)| \leq \frac{C|x - y|^2}{\min(|x|, |y|)^{2+\alpha}}$$

if we use a Taylor inequality. Moreover,

$$\begin{aligned} |K_\varepsilon(x) - K_\varepsilon(y) - \nabla K_\varepsilon(y)(x - y)| &= \int_0^1 ((1 - u)\nabla K_\varepsilon((1 - u)x + uy) - K_\varepsilon(y)) \cdot (x - y) du \\ &\leq \frac{C|x - y|}{\min(|x|, |y|)^{1+\alpha}}. \end{aligned} \quad (3.18)$$

And we can get many inequalities between this two. We fix a positive  $\gamma$  smaller than  $1 - \alpha$ . If we take the first inequality at the power  $\gamma$ , and the second at the power  $(1 - \gamma)$  and multiply them, we obtain

$$|K_\varepsilon(x) - K_\varepsilon(y) - \nabla K_\varepsilon(y)(x - y)| \leq \frac{C|x - y|^{1+\gamma}}{\min(|x|, |y|)^{1+\alpha+\gamma}}. \quad (3.20)$$

Thanks to this inequality, we get that

$$A_2 \geq -\frac{C|X_i(t) - x_t|^{1+\gamma}}{N} \sum_{j \neq i} \frac{1}{(\min(|X_j(t) - x_t|, |X_j(t) - X_i(t)|)^{1+\alpha+\gamma}}$$

We can bound this sum exactly as we do for  $\bar{\nabla}U$ , the derivative of  $u$  (see Lemma 2). The only difference is that  $1 + \alpha$  is replaced by  $1 + \alpha + \gamma$ . We obtain

$$A_2 \geq -C|X_i(t) - x_t|^{1+\gamma} (\|\omega\|_{\infty, \varepsilon} R(t)^{1-\alpha-\gamma} + \varepsilon^{1-\alpha-\gamma} (\|\omega\|_{\infty, \varepsilon} + 2)m(t)^{1+\alpha+\gamma})$$

bound that we will also write that bound  $A_2 \geq -K_2|X_i(t) - x_t|^{1+\gamma}$ , with

$$K_2 = C\|\omega\|_{\infty, \varepsilon}(R(t)^{1-\alpha-\gamma} + m(t)^{1+\alpha+\gamma})$$

Finally, we obtain

$$\frac{d}{dt}|A_t(X_i(t) - x_t)| \geq -\|A_t\|(K_1\varepsilon + K_2|X_i(t) - x_t|^{1+\gamma})$$

Using the remark 4, we can bound  $|X_i(t) - x_t|$  by  $2\rho_t$  so that

$$\frac{d}{dt}|A_t(X_i(t) - x_t)| \geq -(K_1\varepsilon + K_2\rho_t^{1+\gamma}),$$

This is true till  $S_{t'}$  is not too stretched because in that case  $\|A_{t'}\| \leq 2$ . For this, we need to multiply  $K_1$  and  $K_2$  by a numerical constant. And this inequality implies that if  $X_i(t)$  belongs to  $S_t$ , then  $X_{t'}$  will also belongs to  $S_{t'}$ .

There only remains to prove the ODE satisfied by the determinant to finish the proof of this lemma. For this, we just derivate classically the determinant:

$$\frac{d}{ds}(\det(A_s)) = \text{tr}(A_s^{-1}\dot{A}_s) = \text{tr}(\nabla u_\varepsilon(x_s)) = \text{div}(u_\varepsilon(x_s)) = 0,$$

because  $K$  and then  $K_\varepsilon$  are divergence free.  $\square$

Thanks to this lemma, we will get the asymptotic preservation of the  $L^\infty$  discrete norm on the interval of time  $[0, T_1]$ , with  $T_1$  the time so that  $\int_0^{T_1} \bar{\nabla}U(t) dt = 1/4$ . This is the aim of the following proposition:

**Proposition 2.** *We fix a  $\gamma \in (0, 1 - \gamma)$  and a  $T_1$  so that  $\int_0^{T_1} \bar{\nabla}U(t) dt = 1/4$ . Then, there exist two constant  $K_1$  and  $K_2$  depending on  $\gamma$ ,  $R$  and also  $m$  such that for every time  $t \leq T_1$ , we have the following bound on  $\|\omega(t)\|_{\infty, \eta}$ :*

$$\|\omega(t)\|_{\infty, \eta} \leq \|\omega(0)\|_{\infty, \varepsilon} \left(1 + CK_2\eta^\gamma + K_1\frac{\varepsilon}{\eta}\right)^2$$

*Proof.* We choose an  $x$  and denote  $S_t = B(x, \eta)$ . Since,  $B(x, \eta)$  is a not too stretched parallelogram with matrix  $Id$ , we can make it evolve backwards according to the preceding Lemma 6. We obtain a family  $(S_{t'})_{t^* \leq t' < t}$  of parallelograms. The matrix of the family of parallelogram we obtain satisfy  $A'_{t'} = A_{t'}\nabla u_\varepsilon(x_s)$ . This give the bound:

$$\|A_t - Id\| \leq \int_{t'}^t \bar{\nabla}U(s) ds e^{\int_{t'}^s \bar{\nabla}U(s) ds}.$$

Since  $\varepsilon^{1/4}/4 \leq 1/2$ , the condition  $\int_0^t \nabla U(s) ds \leq 1/4$  ensures that  $\|A_t - Id\| \leq 1/2$  up to  $t = 0$ . So it means that the family is define till time  $t^* = 0$ . The parallelogram  $S_0$  will help us to bound  $\omega_N(t, S_t)$  by

$$|\omega_N(t, S_t)| \leq |\omega_N^0(S_0)| \leq \|\omega_N(0)\|_{\infty, \varepsilon} Vol_\varepsilon(S_0)$$

Now, we only need a bound on the size  $\rho_0$  of  $S_0$  to control is  $\varepsilon$ -volume. To be precise, we define  $\kappa'_s = \rho_s - K_1\varepsilon(t-s)$ . With this notation, the ODE satisfied by  $\rho_t$  may be rewritten if  $\rho_s$  is greater than  $K_1\varepsilon$

$$\dot{\kappa}'_s \geq -K_2\kappa'^{1+\gamma}_s.$$

If we integrate this equation, we obtain  $\kappa_s \leq \frac{\rho'_t}{(1 - K_2(t-s)\kappa_t^\gamma)^{1/\gamma}}$ . This inequation give the following one if we replace  $\kappa_u$  by  $\rho_u - K_1\varepsilon(t-u)$  in it:

$$\rho_s \leq \frac{\rho_t}{(1 - K_2(t-s)\rho_t^\gamma)^{1/\gamma}} + K_1(t-s)\varepsilon$$

If  $\rho_t^\gamma T_1$  is choosen sufficiently small, we may rewrite it (with a multiplication by a scalar of  $K_2$ )

$$\rho_s \leq \rho_t(1 + K_2(t-s)\rho_t^\gamma) + K_1(t-s)\varepsilon.$$

So, at time  $t' = 0$ , our vortices were localized in a not too stretched parallelogram of size smaller than

$$\rho_0 \leq \eta(1 + K_2t\eta^\gamma) + K_1t\varepsilon.$$

Using Lemma 5 to control its  $\varepsilon$ -volume, we obtain that

$$Vol(S_0) \leq \eta^2 \left(1 + K_2t\eta^\gamma + K_1\frac{\varepsilon}{\eta}\right)^2$$

Now, if we choose for  $S_t$  every ball of size  $\eta$ , we obtain

$$\|\omega(t)\|_{\infty, \eta} \leq \|\omega(0)\|_{\infty, \varepsilon} \left(1 + K_2t\eta^\gamma + K_1\frac{\varepsilon}{\eta}\right)^2$$

□

The bound we obtain at scale  $\eta$  are much better than the one we obtain at scale  $\varepsilon$ . Thanks to them, the system of inequalities on  $R$  and  $m$  will be sublinear up to some negligible term. The only problem is that this new bound is only valid till the time  $T_1$ . We shall bypass this difficulty by iterating the previous Proposition 2. This give the following Lemma:

**Lemma 7.** *Let  $t$  be a time  $t \in \mathbb{R}^+$  and  $\eta_N$  a sequence of scale going to 0 so that  $\eta/\varepsilon$  goes to  $+\infty$ . We fix a  $N$  and choose a integer  $k$  so that  $\int_0^t \nabla U_N(s) ds \leq k/4$ . We also define*

$$\delta_N = \left(\frac{\varepsilon}{\eta_N}\right)^{1/k}.$$

*Then the following inequality holds if  $N$  is large enough:*

$$\|\omega(t)\|_{\infty, \eta} \leq \|\omega(0)\|_{\infty, \varepsilon} (1 + K_2\eta^\gamma + K_1\delta)^{2k} \quad (3.21)$$

*Proof.* For, the proof, we will erase the subscript  $N$  for clarity, and because we will work at  $N$  constant. To prove the lemma, we only use the proposition 2  $k$  times. The first time between 0 and  $T_1$  with the scale  $\varepsilon$  and  $\varepsilon/\delta$ . The second time between  $T_1$  and  $T_2$ , where  $T_2$  defined by  $\int_{T_1}^{T_2} \nabla U(s) ds \leq 1/4$ , replacing  $\varepsilon$  by  $\varepsilon/\delta$  and  $\varepsilon/\delta$  by  $\varepsilon/\delta^2$  in Proposition 6. We do it  $k$  times, the last time with the scale  $\varepsilon/\delta^{k-1}$  and  $\varepsilon/\delta^k = \eta$  and obtain that:

$$\|\omega(t)\|_{\infty, \eta_k} \leq \|\omega(0)\|_{\infty, \varepsilon} \prod_{l=1}^k \left(1 + K_2\left(\frac{\varepsilon}{\delta^l}\right)^\gamma + K_1\delta\right)^2$$

And this product can be bounded by the right hand side of (3.21). □

Now we can use these bounds on the norms  $\|\omega\|_{\infty,\eta}$  to refine our bound on  $U$  and  $\nabla U$ . Using decomposition of the space at scale  $\eta$  and then at scale  $\varepsilon$  for what is close from the discontinuity, we get the bounds of the following lemma

**Lemma 8.**

$$U(t) \leq C\|\omega\|_{\infty,\eta}^{1/2} R^{1-\alpha} + C\|\omega\|_{\infty,\varepsilon}^{1/2} \eta^{1-\alpha} + \varepsilon^{2-\alpha} \|\omega\|_{\infty,\varepsilon} m(t)^\alpha \quad (3.22)$$

$$\nabla U(t) \leq C\|\omega\|_{\infty,\eta} R^{1-\alpha} + C\|\omega\|_{\infty,\varepsilon} \eta^{1-\alpha} + C\varepsilon^{1-\alpha} (\|\omega\|_{\infty,\varepsilon} + 2) m(t)^{1+\alpha} \quad (3.23)$$

*Proof.* To prove these new estimates, we use exactly the same technic that in Lemma 3.2 and 2. First we use the same decomposition of space, with  $\eta$  replacing  $\varepsilon$ , and then we do a more finer estimate, using  $\varepsilon$ , for the particles  $\eta$  close of our particle, which one we bound the force.  $\square$

### 3.3.2 A new sub-linear system of differential inequalities

We will now introduce a system of differential equation that will be able to give us the bound we need. this is the following one:

$$\begin{cases} \dot{\tilde{m}}(t) = C\|\omega(0)\|_{\infty,\varepsilon} \tilde{m}(t) \tilde{R}(t)^{1-\alpha} + 1 \\ \dot{\tilde{R}}(t) = C\|\omega(0)\|_{\infty,\varepsilon} \tilde{R}(t)^{1-\alpha} + 1 \end{cases}$$

with the initial conditions  $R^0$  and  $m^0$  so that  $m_N(0) \leq m^0$  and  $R_N(0) \leq R^0$  for all  $N$ . We also define

$$\tilde{\nabla} U = C\|\omega(0)\|_{\infty,\varepsilon} \tilde{R}(t)^{1-\alpha},$$

where we use for  $C$  the same constant that in (8). This is a bound for the derivative of the field if we use  $\tilde{R}$  and  $\tilde{m}$  in the bound and neglect the term with power of  $\varepsilon$ .

We now choose a time  $t$ , and fix a  $k$  such that  $\int_0^t \tilde{\nabla} U(s) ds \leq k/4$ . Using the bounds (8), and (3.21) with this  $k$ , we get the following system of inequalities

$$\begin{cases} \dot{R}(t) \leq \|\omega(0)\|_{\infty,\varepsilon}^{1/2} R(t)^{1-\alpha} + S_N^1(R(t), m(t)) \\ \dot{m}(t) \leq m(t) (\|\omega(0)\|_{\infty,\varepsilon} R(t)^{1-\alpha} + S_N^2(R(t), m(t))) \end{cases}$$

where  $S^1$  and  $S^2$  are two polynoms with positive coefficients using  $\|\omega(0)\|_{\infty,\varepsilon}$  containing all a positive power of  $\varepsilon$ , *eta* or  $\varepsilon/\eta$ . This means that they will become small if  $N$  is chosen large enough. More precisely, we choose  $N_0$  such that for  $N \geq N_0$ ,

$$\sup_{s \leq t} \max(S_N^1(\tilde{R}(s), \tilde{m}(s)), S_N^2(\tilde{R}(s), \tilde{m}(s))) \leq \frac{1}{2}.$$

We will show that from this rank,  $m_N$  and  $R_N$  are bounded by  $\tilde{m}$  and  $\tilde{R}$  till time  $t$ . We fix a  $N$  greater than  $N_0$  and define  $\tau_N$  to be the first time where either  $R_N(s) \geq \tilde{R}(t)$  or  $m_N(t) \geq \tilde{m}(t)$ . Till this time,

$$\max_i (S_N^i(R_N(s), m_N(s))(t),) \leq \frac{1}{2},$$

because the  $S_i$  have positive coefficients. Moreover, if  $\tau_N \leq t$ , we have

$$\int_0^{\tau_N} \nabla U_N(s) ds \leq \int_0^t \tilde{\nabla} U(s) ds \leq k/4$$

and then we have the right to use the the bound of Lemma 7, and the system (3.3.2). But at time  $\tau_N$ ,  $\max(S_N^i(R_N(\tau_N), m_N(\tau_N))) \leq 1/2$ . Thanks to this, and the fact (3.3.2), the bounds  $R_N(s) \geq \tilde{R}$  and  $m_N(t) \geq m(t)$  will remains true a little after  $\tau_N$ . so  $\tau_N$  is necessarily greater than  $t$ . And we have the uniform bound we need till  $t$ . The asymptotic preservation of  $\|\omega(t)\|_{\infty,\eta}$  also occurs.

At this point, we only need to use the Lemma 4 to end the proof of the convergence for long time. The argument is the same as for short time, replacing  $\varepsilon$  by  $\eta$ . And the result is proved.

## 4 The vortex system

Why we state result for that kind of kernel and not for true vortices, evolving with the Biot-Savard law? Because in the true case, this technic wouldn't work. Why? Essentially because in the true vortex case, and in the Euler equation, the field of speed is not any more Lipschitz, but satisfy only in the estimate

$$|u(x) - u(y)| \leq -K|x - y| \log(|x - y|).$$

We can obtain some similar estimates for the discrete vector field (see the following lemma 9), but they wouldn't give us the right to conclude as in the previous section. Basically because we won't be able to control  $m$  and the minimal distance between particles any more. This can be seen thank to the following argument: The solution of the EDO  $x' = -x \log(x)$  are of the type  $x(t) = x_0 e^{-t}$ . So, if we look at two particles in the field of a Euler fluid with vorticity in  $L^\infty$ , initially separated by a distance of order  $\varepsilon$ , they only need a time of order 1 to get closer up to a distance of order  $\varepsilon^2$ . And for us, this will imply an explosion of  $m$  in a finite time. For this reason, our technic can not be applied in that case.

In fact, there already exists result of convergence for true vortex system towards true Euler equation. But, they are all relying on the fact that we can use some symetrisation of the Euler kernel, in a little much regular case.

Nevertheless, we state in the following lemma the two estimate we can obtain on  $U$  and its derivative: The second is given in the following lemma

**Lemma 9.** *There exists constants  $C$  such that:*

$$U(t) \leq C \|\omega\|_{\infty, \varepsilon} (R(t) + \varepsilon m(t)) \quad (4.1)$$

$$\sup_{i \neq j} |u(X_i) - u(X_j)| \leq C \|\omega\|_{\infty, \varepsilon} ((1 + \log(R) + \log(\max(\varepsilon, |X_i - X_j|))) |X_i - X_j| + \varepsilon m(t)) \quad (4.2)$$

*Proof of the lemma.* We choose a couple  $(i, j)$  to estimate the difference  $|u(X_i) - u(X_j)|$ . We will of course decompose the space in cells of size  $\varepsilon$ . We pick one integer  $L$  depending on  $\varepsilon$  that we will fix later and decompose the sum

$$\begin{aligned} |u(X_i) - u(X_j)| &\leq \frac{1}{N} \sum_{\min(|X_k - X_i|, |X_k - X_j|) \geq 2^L \varepsilon} \frac{|X_i - X_j|}{\min(|X_k - X_i|, |X_k - X_j|)^2} \\ &\quad + \frac{2}{N} \sum_{\varepsilon \leq \min(|X_k - X_i|, |X_k - X_j|) \leq 2^L \varepsilon} \frac{1}{\min(|X_k - X_i|, |X_k - X_j|)} \\ &\quad + \frac{2}{N} \sum_{\min(|X_k - X_i|, |X_k - X_j|) \leq \varepsilon} \frac{1}{\min(|X_k - X_i|, |X_k - X_j|)} \end{aligned} \quad (4.3)$$

We decompose the first sum  $\bar{\nabla} U_1$  in

$$\bar{\nabla} U_1 = \frac{1}{N} \sum_{l=1}^{l_{max}} \sum_{k \in I_l} \frac{|X_i - X_j|}{\min(|X_k - X_i|, |X_k - X_j|)^2}$$

where  $I_l = \{k | \min(|X_k - X_i|, |X_k - X_j|) \in [2^{L+l}\varepsilon, 2^{L+l+1}\varepsilon]\}$  and  $l_{max} = \lceil \log_2(R/2^L\varepsilon) \rceil + 1$ . The total vorticity in  $I_k$  can be bounded  $\omega_N(I_k) \leq C(2^{L+l}\varepsilon)^2 \|\omega\|_{\infty, \varepsilon}$ . We get:

$$\bar{\nabla} U_1 \leq |X_i - X_j| \sum_{l=1}^{l_{max}} \frac{C \|\omega\|_{\infty, \varepsilon} (2^{L+l}\varepsilon)^2}{(2^{L+l}\varepsilon)^2} \quad (4.4)$$

$$\leq C \|\omega\|_{\infty, \varepsilon} |X_i - X_j| \sum_{l=1}^{l_{max}} 1 \quad (4.5)$$

As  $l_{max} = \log(R/2^L\varepsilon)$ , we get

$$\bar{\nabla} U_1 \leq C \|\omega\|_{\infty, \varepsilon} |X_i - X_j| \log\left(\frac{R}{2^L\varepsilon}\right)$$

For the second sum  $\bar{\nabla} U_2$ , we use the decomposition  $J_m = \{k | \min(|X_k - X_i|, |X_k - X_j|) \in [2^m\varepsilon, 2^{m+1}\varepsilon]\}$ , for  $m = 1$  to  $L-1$ . The total vorticity in  $J_m$  satisfy  $\omega(J_m) \leq \|\omega\|_{\infty, \varepsilon} (2^m\varepsilon)^2$ . So,

$$\bar{\nabla} U_2 \leq C \|\omega\|_{\infty, \varepsilon} \sum_{m=1}^{L-1} \frac{(2^m\varepsilon)^2}{2^m\varepsilon} \leq C \|\omega\|_{\infty, \varepsilon} 2^L\varepsilon$$

And the last term  $\bar{\nabla} U_3$  is bounded by

$$\bar{\nabla} U_3 \leq 2 \|\omega\|_{\infty, \varepsilon} \varepsilon m(t)$$

Putting the three bound together, we obtain

$$\bar{\nabla} U \leq C \|\omega\|_{\infty, \varepsilon} |X_i - X_j| \log\left(\frac{R}{2^L\varepsilon}\right) + C \|\omega\|_{\infty, \varepsilon} (2^L + m(t)) \quad (4.6)$$

What is the best choice to do for  $L$ ? It is to take  $\varepsilon 2^{-L} = |X_i - X - J|$ , if  $|X_i - X - J| \geq \varepsilon$  as in the continuous case. Of course, we can not obtain exact equality, but only equality up to a factor two, because  $L$  must be an integer. But this factor two will not raise any difficulty and will disappear in the constant  $C$ . If  $|X_i - X - J| < \varepsilon$  the best choice is to take  $L = 1$ . So, with choice, we obtain:

$$\sup_{i \neq j} |u(X_i) - u(X_j)| \leq C \|\omega\|_{\infty, \varepsilon} ((1 + \log(R) + \log(\max(\varepsilon, |X_i - X_j|))) |X_i - Y_j| + \varepsilon m(t)) \quad (4.7)$$

□

With this lemma, we are not able to conclude, because the bound on  $\nabla \bar{U}$  do not allow us to bound the grow of  $m$ .

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